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Systems of coherent vectors

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Abstract. Some remarkable overcomplete sytems of vectors (called *systems of coherent vectors*), which offer almost the same facilities as an orthonormal basis and a simpler description of the action of the group, are defined in the case of an orthogonal $\mathbb{R}$-irreducible representation of a finite group G by following the analogy with the theory of coherent states.

1. Introduction

A vector space is usually described by using a basis, but certain overcomplete systems of vectors may also lead to useful descriptions. The method is mainly used in the case of infinite-dimensional spaces where the coherent states are such systems of vectors [5, 6, 8, 10, 12]. Among the applications in the case of finite-dimensional spaces there is a three-axis description of the honeycomb lattice [1, 11], a four-axis description of the diamond structure [2, 3] and the description of some $\mathbb{Z}$-modules in quasicrystal physics [4, 7, 9].

New applications of this method in the case of finite-dimensional real spaces seem to be possible, and our purpose is to develop the corresponding general mathematical formalism. Some of our results are inspired by the theory of coherent states [8, 10], but we think that certain constructions done in the simpler case of a finite-dimensional space might also offer some suggestions for the theory of coherent states.

2. Systems of coherent vectors

Let G be a finite group, $g : \mathbb{E}_n \longrightarrow \mathbb{E}_n$ be an orthogonal $\mathbb{R}$-irreducible representation of G in the Euclidean space $\mathbb{E}_n = (\mathbb{R}^n, \langle\, ,\, \rangle)$, where $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$, and let $u_1 \in \mathbb{E}_n$ be a fixed non-zero vector. The set H of all the elements $g \in G$ such that $gu_1 \in \{u_1, -u_1\}$ is a subgroup of G. Consider the space $M = G/H$ of all the left cosets of G corresponding to H, and fix a set $\{g_1, g_2, \ldots, g_k\} \subset G$ containing one and only one representative of each coset. We can assume that g_1 is the unit element of G.

Let $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ be the vectors of the canonical basis of $\mathbb{E}_n$, and let $\mathcal{C} = \{u_1, u_2, \ldots, u_k\}$, where $u_i = g_i u_1$. Since $g_i u_1 = g_j u_1 \Longrightarrow g_i^{-1} g_j \in H$, that is, $u_i = u_j \Longrightarrow g_j \in g_i H$, the mapping

$$\Phi : M \longrightarrow \mathcal{C} \qquad \Phi(g_i) = u_i \tag{1}$$

is a bijection which allows us to identify the sets M and $\mathcal{C}$. If g_i is replaced by another representative g_i' of the coset $g_i H$ then $g_i' u_1 \in \{u_i, -u_i\}$.

For each $g \in G$ there exist the numbers $s_1^g, s_2^g, \ldots, s_k^g \in \{-1, 1\}$ and a permutation of the set $\{1, 2, \ldots, k\}$ denoted also by g such that

$$gu_i = s_{g(i)}^g u_{g(i)} \tag{2}$$

for all $i \in \{1, 2, \ldots, k\}$, and hence the subspace of $\mathbb{E}_n$ generated by $\mathcal{C}$ is G-invariant. Since the considered representation of G in $\mathbb{E}_n$ is $\mathbb{R}$-irreducible this subspace must coincide with $\mathbb{E}_n$, that is, $\mathcal{C}$ contains a basis of $\mathbb{E}_n$ (in the following, we shall consider only the case when $\mathcal{C}$ is not itself a basis). We shall show that such a system leads to a useful description of $\mathbb{E}_n$, and call it a *system of coherent vectors* (SCV).

Theorem 1. *There is a constant $\kappa \in (0, \infty)$ such that*

$$x = \kappa \sum_{i=1}^{k} \langle x, u_i\rangle u_i = \kappa \sum_{y\in\mathcal{C}} \langle x, y\rangle y \tag{3}$$

for all $x \in \mathbb{E}_n$. In addition,

$$\langle x, y\rangle = \kappa \sum_{i=1}^{k} \langle x, u_i\rangle\langle u_i, y\rangle \qquad \|x\|^2 = \kappa \sum_{i=1}^{k} \langle x, u_i\rangle^2 \tag{4}$$

for all $x, y \in \mathbb{E}_n$.

Proof. The operator $U : \mathbb{E}_n \longrightarrow \mathbb{E}_n$, $Ux = \sum_{i=1}^{k} \langle x, u_i\rangle u_i$ satisfies the relation

$$U(gx) = \sum_{i=1}^{k} \langle gx, u_i\rangle u_i = \sum_{i=1}^{k} \langle gx, gu_i\rangle gu_i = g\left(\sum_{i=1}^{k} \langle x, u_i\rangle u_i\right) = g(Ux)$$

for all $x \in \mathbb{E}_n$, $g \in G$. Since our representation is in a real space, we cannot use Schur's lemma in order to obtain directly that U must have the form $Ux = \lambda x$. However, a real version of this lemma holds since U is a self-adjoint operator

$$\langle Ux, y\rangle = \left\langle \sum_{i=1}^{k} \langle x, u_i\rangle u_i, y \right\rangle = \sum_{i=1}^{k} \langle x, u_i\rangle\langle u_i, y\rangle = \langle x, Uy\rangle$$

and, hence, it has a real eigenvalue λ. Denoting $V = \{x \in \mathbb{E}_n | Ux = \lambda x\}$, from the relation $Ux = \lambda x \Longrightarrow U(gx) = \lambda(gx)$, it follows $g(V) \subset V$, and in view of the irreducibility of the representation of G in $\mathbb{E}_n$ we must have $V = \mathbb{E}_n$; that is, $Ux = \lambda x$ for all $x \in \mathbb{E}_n$. From $\lambda e_1 = Ue_1 = \sum_{i=1}^{k} \langle e_1, u_i\rangle u_i$ we obtain $\lambda = \langle \lambda e_1, e_1\rangle = \sum_{i=1}^{k} \langle e_1, u_i\rangle^2 \in (0, \infty)$, and hence there is $\kappa \in (0, \infty)$ such that $\lambda = 1/\kappa$. For any $y \in \mathbb{E}_n$ we obtain

$$\langle x, y\rangle = \left\langle \kappa \sum_{i=1}^{k} \langle x, u_i\rangle u_i, y \right\rangle = \kappa \sum_{i=1}^{k} \langle x, u_i\rangle\langle u_i, y\rangle$$

whence $\|x\|^2 = \kappa \sum_{i=1}^{k} \langle x, u_i\rangle^2$. □

Using the notations $|g\rangle = \Phi(g)$, $\langle g|x\rangle = \langle \Phi(g), x\rangle$, the relation (3) can be rewritten in the form $x = \kappa \sum_{g\in M} \langle g|x\rangle |g\rangle$, in agreement with the theory of coherent states [10, relation (16)].

Example 1. The relations

$$a(x_1, x_2, x_3) = (-x_2, x_1, -x_3) \qquad b(x_1, x_2, x_3) = (x_1, -x_3, -x_2) \tag{5}$$

define an orthogonal $\mathbb{R}$-irreducible representation of the complete tetrahedral group $T_d = \bar{4}3m = \langle a, b | a^4 = b^2 = (ab)^3 = e\rangle$ in $\mathbb{E}_3$. Choosing $u_1 = (1, 1, 1)$ we obtain the SCV

$$\mathcal{C} = \{(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\}. \tag{6}$$

One can also remark that the SCV obtained by starting from $u_1 = (1, 0, 0)$ coincides with the canonical basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

Example 2. Let $\tau = (1+\sqrt{5})/2$. The relations

$$\begin{aligned} a(x_1,x_2,x_3) = \bigg(& \frac{\tau-1}{2}x_1 - \frac{\tau}{2}x_2 + \frac{1}{2}x_3,\ \frac{\tau}{2}x_1 + \frac{1}{2}x_2 \\ & + \frac{\tau-1}{2}x_3,\ -\frac{1}{2}x_1 + \frac{\tau-1}{2}x_2 + \frac{\tau}{2}x_3 \bigg) \qquad (7) \\ b(x_1,x_2,x_3) = {} & (-x_1,-x_2,x_3) \end{aligned}$$

define an orthogonal $\mathbb{R}$-irreducible representation of the icosahedral group $Y = 235 = \langle a,b | a^5 = b^2 = (ab)^3 = e\rangle$ in $\mathbb{E}_3$. Choosing $u_1 = (0,1,\tau)$ we obtain [8] the SCV

$$\mathcal{C} = \{(0,1,\tau),(1,\tau,0),(-1,\tau,0),(-\tau,0,1),(0,-1,\tau),(\tau,0,1)\} \qquad (8)$$

and choosing $u_1 = (1,1,1)$ we obtain the SCV $\mathcal{C} = \{u_1,u_2,\ldots,u_{10}\}$, where

$$\begin{array}{lll} u_2 = (-1,1,1) & u_5 = (0,\tau,\tau-1) & u_8 = (0,\tau,1-\tau) \\ u_3 = (1,-1,1) & u_6 = (\tau-1,0,\tau) & u_9 = (1-\tau,0,\tau) \\ u_4 = (1,1,-1) & u_7 = (\tau,\tau-1,0) & u_{10} = (\tau,1-\tau,0). \end{array} \qquad (9)$$

3. A geometric interpretation

A geometric interpretation of our formalism can be obtained by using some mathematical results obtained in connection with the strip projection method [4, 8].

Lemma 1. *The relation*

$$T_g(\alpha_1,\alpha_2,\ldots,\alpha_k) = (s_1^g\alpha_{g^{-1}(1)}, s_2^g\alpha_{g^{-1}(2)},\ldots,s_k^g\alpha_{g^{-1}(k)}) \qquad (10)$$

defines an orthogonal representation of G in $\mathbb{E}_k$.

Proof. Since $s_{(hg)(i)}^{hg}u_{(hg)(i)} = (hg)u_i = h(s_{g(i)}^g u_{g(i)}) = s_{g(i)}^g h u_{g(i)} = s_{g(i)}^g s_{(hg)(i)}^h u_{(hg)(i)}$ we obtain $s_{(hg)(i)}^{hg} = s_{g(i)}^g s_{(hg)(i)}^h$, that is, $s_i^{hg} = s_{h^{-1}(i)}^g s_i^h$, whence

$$\begin{aligned} T_h(T_g(\alpha_1,\ldots,\alpha_k)) &= T_h(s_1^g\alpha_{g^{-1}(1)},\ldots,s_k^g\alpha_{g^{-1}(k)}) \\ &= (s_1^h s_{h^{-1}(1)}^g \alpha_{g^{-1}(h^{-1}(1))},\ldots,s_k^h s_{h^{-1}(k)}^g\alpha_{g^{-1}(h^{-1}(k))}) \\ &= (s_1^{hg}\alpha_{(hg)^{-1}(1)},\ldots,s_k^{hg}\alpha_{(hg)^{-1}(k)}) = T_{hg}(\alpha_1,\ldots,\alpha_k). \end{aligned}$$

In addition, $\langle T_g\alpha, T_g\beta\rangle = \sum_{i=1}^k (s_i^g)^2\alpha_{g^{-1}(i)}\beta_{g^{-1}(i)} = \sum_{i=1}^k \alpha_i\beta_i = \langle\alpha,\beta\rangle$. □

Lemma 2. *The subspaces*

$$\mathbb{E}_k^{\|} = \{(\langle x,u_1\rangle,\langle x,u_2\rangle,\ldots,\langle x,u_k\rangle) | x\in\mathbb{E}_n\} \qquad (11)$$

$$\mathbb{E}_k^{\perp} = \left\{(\alpha_1,\alpha_2,\ldots,\alpha_k)\in\mathbb{E}_k \,\middle|\, \sum_{i=1}^k \alpha_i u_i = 0\right\} \qquad (12)$$

are orthogonal, G-invariant, and $\mathbb{E}_k = \mathbb{E}_k^{\|}\oplus\mathbb{E}_k^{\perp}$.

Proof. Indeed, $\dim\mathbb{E}_k^{\|} = n$, $\dim\mathbb{E}_k^{\perp} = k-n$, and

$$\begin{aligned} T_g(\langle x,u_1\rangle,\ldots,\langle x,u_k\rangle) &= (s_1^g\langle x,u_{g^{-1}(1)}\rangle,\ldots,s_k^g\langle x,u_{g^{-1}(k)}\rangle) \\ &= (\langle x,g^{-1}u_1\rangle,\ldots,\langle x,g^{-1}u_k\rangle) = (\langle gx,u_1\rangle,\ldots,\langle gx,u_k\rangle) \\ \sum_{i=1}^k s_i^g\alpha_{g^{-1}(i)}u_i = \sum_{i=1}^k \alpha_i s_{g(i)}^g u_{g(i)} &= \sum_{i=1}^k \alpha_i g u_i = g\left(\sum_{i=1}^k\alpha_i u_i\right) = 0 \qquad (13) \\ \sum_{i=1}^k \langle x,u_i\rangle\alpha_i &= \left\langle x,\sum_{i=1}^k\alpha_i u_i\right\rangle = 0 \end{aligned}$$

for any $g \in G$, $x \in \mathbb{E}_n$, and $(\alpha_1, \dots, \alpha_k) \in \mathbb{E}_k^{\perp}$. □

Lemma 3. *The orthogonal projector corresponding to* $\mathbb{E}_k^{\parallel}$ *is*

$$\pi : \mathbb{E}_k \longrightarrow \mathbb{E}_k \qquad \pi\alpha = \left(\kappa \sum_{i=1}^{k} \langle u_1, u_i\rangle \alpha_i, \dots, \kappa \sum_{i=1}^{k} \langle u_k, u_i\rangle \alpha_i\right). \tag{14}$$

Proof. If $\alpha = (\langle x, u_1\rangle, \dots, \langle x, u_k\rangle) \in \mathbb{E}_k^{\parallel}$ then

$$\kappa \sum_{i=1}^{k} \langle u_j, u_i\rangle \alpha_i = \kappa \sum_{i=1}^{k} \langle u_j, u_i\rangle \langle x, u_i\rangle = \langle x, u_j\rangle$$

for all $j \in \{1, 2, \dots, k\}$, that is, $\pi\alpha = \alpha$. For $\alpha \in \mathbb{E}_k^{\perp}$, from the relation

$$\kappa \sum_{i=1}^{k} \langle u_j, u_i\rangle \alpha_i = \kappa \left\langle u_j, \sum_{i=1}^{k} \alpha_i u_i \right\rangle = 0$$

satisfied for all $j \in \{1, 2, \dots, k\}$, it follows that $\pi\alpha = 0$. □

Lemma 4. *The isometry*

$$S : \mathbb{E}_n \longrightarrow \mathbb{E}_k^{\parallel} \qquad Sx = (\sqrt{\kappa}\langle x, u_1\rangle, \dots, \sqrt{\kappa}\langle x, u_k\rangle) \tag{15}$$

is an isomorphism of representations which allows us to identify the spaces $\mathbb{E}_n$ *and* $\mathbb{E}_k^{\parallel}$.

Proof. The mapping S is an isometry

$$\langle Sx, Sy\rangle = \kappa \sum_{i=1}^{k} \langle x, u_i\rangle \langle y, u_i\rangle = \langle x, y\rangle$$

and in view of (13) we have $T_g(Sx) = S(gx)$ for any $x \in \mathbb{E}_n$, $g \in G$. □

The representation of a vector $x \in \mathbb{E}_n$ as a linear combination of $u_1, u_2, \dots, u_k$ is not unique, but the next theorem shows that the canonical representation (3) is a privileged one, namely, the sum of the square of the coefficients takes its minimal value.

Theorem 2. *Let* $x \in \mathbb{E}_n$. *If* $x = \sum_{i=1}^{k} \alpha_i u_i$ *then*

$$\begin{aligned} &(i) \quad \pi(\alpha_1, \dots, \alpha_k) = (\kappa\langle x, u_1\rangle, \dots, \kappa\langle x, u_k\rangle) \\ &(ii) \quad \sum_{i=1}^{k} \alpha_i^2 \geqslant \sum_{i=1}^{k} (\kappa\langle x, u_i\rangle)^2. \end{aligned} \tag{16}$$

Particularly,

$$\pi(\kappa\langle x, u_1\rangle, \dots, \kappa\langle x, u_k\rangle) = (\kappa\langle x, u_1\rangle, \dots, \kappa\langle x, u_k\rangle). \tag{17}$$

Proof. In view of lemma 3 we obtain

$$(\pi\alpha)_i = \kappa \sum_{j=1}^{k} \langle u_i, u_j\rangle \alpha_j = \kappa \left\langle u_i, \sum_{j=1}^{k} \alpha_j u_j \right\rangle = \kappa\langle u_i, x\rangle.$$

Since the vectors $\pi\alpha$ and $\alpha - \pi\alpha$ are orthogonal, we obtain

$$\sum_{i=1}^{k} \alpha_i^2 = \|\alpha\|^2 = \|\pi\alpha\|^2 + \|\alpha - \pi\alpha\|^2 \geqslant \|\pi\alpha\|^2 = \sum_{i=1}^{k} (\kappa\langle x, u_i\rangle)^2.$$

□

The numbers $\kappa\langle x, u_1\rangle, \dots, \kappa\langle x, u_k\rangle$ are the *coordinates* of $x \in \mathbb{E}_n$ with respect to the SCV $\{u_1, \dots, u_k\}$. One can remark that $\alpha_1, \dots, \alpha_k$ are the coordinates of a vector $x \in \mathbb{E}_n$ with respect to $\{u_1, \dots, u_k\}$ if and only if $\pi(\alpha_1, \dots, \alpha_k) = (\alpha_1, \dots, \alpha_k)$, that is, if and only if

$$\alpha_i = \kappa \sum_{j=1}^{k} \langle u_i, u_j\rangle \alpha_j \tag{18}$$

for all $i \in \{1, 2, \dots, k\}$.

The space $\mathbb{E}_k$ can be identified with the space of all the functions $\varphi : M \longrightarrow \mathbb{R}$ by associating the function $\varphi_\alpha(g_i) = \alpha_i$ with each $\alpha \in \mathbb{E}_k$. In this case, denoting $\langle h|g\rangle = \langle \Phi(h), \Phi(g)\rangle$, relation (18) becomes

$$\varphi(h) = \kappa \sum_{g\in M} \langle h|g\rangle \varphi(g) \tag{19}$$

in agreement with the theory of coherent states [10, relation (14)].

4. Description of linear operators in terms of an SCV

It is well known that an adequate description can simplify the solution of a problem. If the elements of $\mathbb{E}_n$ are described by using their coordinates with respect to an SCV $\mathcal{C}$ then each transformation $g : \mathbb{E}_n \longrightarrow \mathbb{E}_n$ becomes a signed permutation

$$T_g(\alpha_1, \alpha_2, \dots, \alpha_k) = (s_1^g \alpha_{g^{-1}(1)}, s_2^g \alpha_{g^{-1}(2)}, \dots, s_k^g \alpha_{g^{-1}(k)})$$

and, consequently, the form of the G-invariant mathematical objects (operators, equations, etc) defined on $\mathbb{E}_n$ becomes simpler. Some concrete applications of the description of $\mathbb{E}_3$ obtained by using the SCV $\{(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\}$ can be found in [2, 3]. In this section we only present some general results concerning the description of the linear operators in terms of an SCV.

Let $A : \mathbb{E}_n \longrightarrow \mathbb{E}_n$ be a linear operator, $\tilde{S}$ be the transpose of the matrix

$$S = \sqrt{\kappa}\begin{pmatrix} \langle u_1, e_1\rangle & \cdots & \langle u_1, e_n\rangle \\ \cdots & \cdots & \cdots \\ \langle u_k, e_1\rangle & \cdots & \langle u_k, e_n\rangle \end{pmatrix} \qquad \pi = \kappa\begin{pmatrix} \langle u_1, u_1\rangle & \cdots & \langle u_1, u_k\rangle \\ \cdots & \cdots & \cdots \\ \langle u_k, u_1\rangle & \cdots & \langle u_k, u_k\rangle \end{pmatrix}$$

and let

$$A = \begin{pmatrix} \langle e_1, Ae_1\rangle & \cdots & \langle e_1, Ae_n\rangle \\ \cdots & \cdots & \cdots \\ \langle e_n, Ae_1\rangle & \cdots & \langle e_n, Ae_n\rangle \end{pmatrix} \qquad A' = \kappa\begin{pmatrix} \langle u_1, Au_1\rangle & \cdots & \langle u_1, Au_k\rangle \\ \cdots & \cdots & \cdots \\ \langle u_k, Au_1\rangle & \cdots & \langle u_k, Au_k\rangle \end{pmatrix}$$

be the matrices associated with our operator with respect to the canonical basis $\{e_1, \dots, e_n\}$ and the SCV $\mathcal{C} = \{u_1, \dots, u_k\}$, respectively.

Theorem 3. *We have*

$$S\tilde{S} = \pi \qquad \tilde{S}S = I \qquad A' = SA\tilde{S} \qquad \mathrm{Tr}\, A = \mathrm{Tr}\, A' \tag{20}$$

where I is the unit matrix.

Proof. Indeed,

$$\kappa \sum_{m=1}^{n} \langle u_i, e_m\rangle \langle u_j, e_m\rangle = \kappa\langle u_i, u_j\rangle$$

$$\kappa \sum_{m=1}^{k} \langle u_m, e_i \rangle \langle u_m, e_j \rangle = \langle e_i, e_j \rangle$$

$$\kappa \sum_{j=1}^{n} \sum_{m=1}^{n} \langle u_i, e_j \rangle \langle e_j, Ae_m \rangle \langle u_l, e_m \rangle = \kappa \langle u_i, Au_l \rangle$$

and

$$\operatorname{Tr} A' = \kappa \sum_{i=1}^{k} \langle u_i, Au_i \rangle = \kappa \sum_{i=1}^{k} \sum_{j=1}^{n} \sum_{m=1}^{n} \langle u_i, e_j \rangle \langle e_j, Ae_m \rangle \langle u_i, e_m \rangle$$

$$= \sum_{j=1}^{n} \sum_{m=1}^{n} \langle e_m, e_j \rangle \langle e_j, Ae_m \rangle = \sum_{j=1}^{n} \langle e_j, Ae_j \rangle = \operatorname{Tr} A.$$

□

Theorem 4. *The operator $A : \mathbb{E}_n \longrightarrow \mathbb{E}_n$ is a self-adjoint operator if and only if the matrix A' is symmetric.*

Proof. If A is self-adjoint then $\langle u_l, Au_m \rangle = \langle u_m, Au_l \rangle$ for all $l, m \in \{1, 2, \ldots, k\}$. Conversely, if this relation is satisfied, then

$$\langle e_i, Ae_j \rangle = \kappa^2 \sum_{l=1}^{k} \sum_{m=1}^{k} \langle e_i, u_l \rangle \langle u_l, Au_m \rangle \langle e_j, u_m \rangle$$

$$= \kappa^2 \sum_{l=1}^{k} \sum_{m=1}^{k} \langle e_i, u_l \rangle \langle u_m, Au_l \rangle \langle e_j, u_m \rangle = \langle e_j, Ae_i \rangle.$$

□

Theorem 5. *Any eigenvalue of A is at the same time an eigenvalue of the matrix A'.*

Proof. If $x \in \mathbb{E}_n$ is an eigenvector corresponding to λ then

$$A'(Sx) = (SA\tilde{S})(Sx) = S(Ax) = S(\lambda x) = \lambda(Sx)$$

that is, Sx is an eigenvector of A' corresponding to the eigenvalue λ. □

Theorem 6. *If $A, B : \mathbb{E}_n \longrightarrow \mathbb{E}_n$ are two linear operators such that $AB = BA$, and A' and B' are the corresponding matrices with respect to $\mathcal{C}$, then $A'B' = B'A'$.*

Proof. We have $A'B' = (SA\tilde{S})(SB\tilde{S}) = S(AB)\tilde{S} = S(BA)\tilde{S} = B'A'$. □

The derivative of a function $f : \mathbb{E}_n \longrightarrow \mathbb{R}$ with respect to a unit vector $v \in \mathbb{E}_n$ is usually defined as

$$\frac{\partial f}{\partial v}(x) = \lim_{t \to 0} \frac{1}{t}(f(x + tv) - f(x)).$$

If $\|u_1\| = 1$ then $\|u_2\| = \cdots = \|u_k\| = 1$, and we have the relations

$$\frac{\partial}{\partial u_i} = \sum_{j=1}^{n} \langle u_i, e_j \rangle \frac{\partial}{\partial e_j} \qquad \frac{\partial}{\partial e_i} = \kappa \sum_{j=1}^{k} \langle e_i, u_j \rangle \frac{\partial}{\partial u_j} \tag{21}$$

which are useful in the case of differential equations and operators.

The description obtained by using an SCV can be regarded as a discrete version of the method of coherent states, and it may be a useful alternative to the usual description in certain applications. Generally, the use of an SCV leads to simpler expressions for the G-invariant objects, and hence a simplification of the mathematical formalism of certain models [2, 3].

References

[1] Burns G and Glazer A M 1978 *Space Groups for Solid State Scientists* (New York: Academic) pp 30–1
[2] Cotfas N 1991 Diamond structure in a natural description *J. Phys.: Condens. Matter* **3** 9279–96
[3] Cotfas N 1995 A mathematical model for diamond type crystals *J. Phys. A: Math. Gen.* **28** 1371–9
[4] Cotfas N 1999 Permutation representations defined by G-clusters with applications to quasicrystals *Lett. Math. Phys.* **47** 111–23
[5] Gazeau J P and Klauder J R 1999 Coherent states for systems with discrete and continuous spectrum *J. Phys. A: Math. Gen.* **32** 123–32
[6] Glauber R J 1963 Coherent and incoherent states of the radiation field *Phys. Rev.* **131** 2766–88
[7] Katz A and Duneau M 1986 Quasiperiodic patterns and icosahedral symmetry *J. Physique* **47** 181–96
[8] Klauder J R 1963 Continuous-representation theory I. Postulates of continuous-representation theory *J. Math. Phys.* **4** 1058–61
[9] Moody R V 1997 Meyer sets and their duals *The Mathematics of Long-Range Aperiodic Order* ed R V Moody (Dordrecht: Kluwer) pp 403–41
[10] Perelomov A M 1972 Coherent states for arbitrary Lie group *Commun. Math. Phys.* **26** 222–36
[11] Sirotin Yu I and Shaskolskaya M P 1982 *Fundamentals of Crystal Physics* (Moscow: Mir) pp 98–101
[12] Zhang W-M, Feng D H and Gilmore R 1990 Coherent states: theory and some applications *Rev. Mod. Phys.* **62** 867–9270